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Strong Uniqueness for Chebyshev Approximation by Reciprocals of Polynomials on $[0, \infty)$

DARRELL SCHMIDT

Department of Mathematical Sciences, Oakland University, Rochester, Michigan 48063

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1. INTRODUCTION

Let $C_0^+[0, \infty) = \{f \in C[0, \infty): f(x) > 0$ for $x \in [0, \infty)$ and $\lim_{x \to \infty} f(x) = 0\}$ and $R_n = \{1/p: p \in \Pi_n$ and $p(x) > 0$ for $x \in [0, \infty)\}$, where Π_n denotes the set of all real algebraic polynomials of degree n or less. For $g \in C_0^+[0, \infty)$, define $\| g\| = \sup\{| g(x)|: x \in [0, \infty)\}$. Uniform approximation of functions in $C_0^+[0, \infty)$ by reciprocals of polynomials has been the topic of several recent investigations (see [1, 2, 5] and the references of [5]). In this setting, it is known that best approximations exist and are unique [1, 2] and the following characterization theorem holds [2].

THEOREM 1. *Let $f \in C_0^+[0, \infty)\backslash R_n$ with $n \geqslant 1$, and let $1/p^* \in R_n$. Then $1/p^*$ is a best approximation to f from R_n if and only if*

(i) (standard alternation) *there exist $n+2$ points $0 \leqslant x_0 < x_1 < \cdots < x_{n+1}$ such that $|f(x_i) - 1/p^*(x_i)| = \|f - 1/p^*\|$, $i = 0,..., n+1$, and $f(x_i) - 1/p^*(x_i) = -(f(x_{i+1}) - 1/p^*(x_{i+1}))$, $i = 0,..., n$, or*

(ii) (nonstandard alternation) *$\partial p^* \leqslant n-1$ and there exist $n+1$ points $0 \leqslant x_0 < x_1 < \cdots < x_n$ such that $f(x_i) - 1/p^*(x_i) = (-1)^{n-i} \|f - 1/p^*\|$, $i = 0,..., n$, where ∂p^* denotes the degree of p^*.*

In the remainder of this note $1/p_f$ shall denote the best approximation to $f \in C_0^+[0, \infty)$ from R_n.

In this note we study strong uniqueness of $1/p_f$ and point Lipschitz continuity of the best approximation operator. In Brink [1], it was shown that if $\partial p_f = n$, then $1/p_f$ is a strongly unique best approximation to f from R_n, that is, there is a constant $\gamma_f > 0$ such that

$$\|f - 1/p\| \geqslant \|f - 1/p_f\| + \gamma_f \| 1/p - 1/p_f\|$$

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for all $1/p \in R_n$. A companion result is that $1/p_f$ is point Lipschitz continuous at f, that is, there is a constant $\lambda_f > 0$ such that

$$\|1/p_g - 1/p_f\| \leqslant \lambda_f \|g - f\|$$

for all $g \in C_0^+[0, \infty)$. In Dunham and Taylor [5] it was shown that if $\partial p_f < n-1$, then $1/p_f$ is discontinuous at f. Thus if $\partial p_f < n-1$, neither strong uniqueness nor point Lipschitz continuity can hold. In [5], however, it was proven that $1/p_f$ is continuous at f if $\partial p_f = n-1$. Left open are the questions of strong uniqueness and point Lipschitz continuity of $1/p_f$ at f when $\partial p_f = n-1$. In this note, we show that if $\partial p_f = n-1$, then $1/p_f$ is strongly unique and point Lipschitz continuous at f.

2. Preliminary Results

The lemmas in this section are motivated by a characterization of strong unicity constants for polynomial approximation due to Cline [4]. The results of this section will subsequently be used to establish the strong unicity result for best reciprocal approximation.

Let $0 \leqslant x_0 < x_1 \cdots < x_n$ be fixed, and for $t > x_n$, let

$$Q_t = \{p \in \Pi_n : (-1)^{n-i} p(x_i) \leqslant 1, \quad i = 0,..., n, \text{ and } p(t) \geqslant -1\}. \tag{2.1}$$

By the assertion on p. 64 of Rice [6] (Note the misprint: = should be replaced by $\geqslant$.) and the fact that all norms on a finite dimensional vector space are equivalent, Q_t is a compact subset of Π_n, where Π_n carries the Euclidean norm of the coefficient vectors. For $x \geqslant 0$, let

$$M(t, x) = \max\{|p(x)| : p \in Q_t\}. \tag{2.2}$$

We express $M(t, x)$ in terms of $n+2$ interpolating polynomials. Define $q \in \Pi_n$ by

$$q(x_i) = (-1)^{n-i}, \quad i = 0,..., n,$$

and for $j = 0,..., n$, define $q_j(t, \cdot) \in \Pi_n$ by

$$q_j(t, x_i) = (-1)^{n-i}, \quad i = 0,..., n, \ i \neq j,$$

and $q_j(t, t) = -1$.

Lemma 2. *For* $0 \leqslant x_0 < x_1 < \cdots < x_n < t$ *and* $x \geqslant 0$, $M(t, x) = \max\{|q(x)|; |q_j(t, x)|, j = 0,..., n\}$.

Proof. Write $t = x_{n+1}$ and note that $p(t) \geqslant -1$ is equivalent to $(-1)^{n-(n+1)} p(x_{n+1}) \leqslant 1$. If $(-1)^{n-(n+1)} p(x_{n+1}) > 0$, q would have $n+1$ zeros. Thus $(-1)^{n-(n+1)} q(x_{n+1}) \leqslant 1$ and $q \in Q_t$. Similarly, $q_j(t, \cdot) \in Q_t$, $j = 0,..., n$. Thus $M(t, x) \geqslant \max\{|q(x)|; |q_j(t, x)|, j = 0,..., n\}$.

Let $p \in Q_t$ satisfy $|p(x)| = M(t, x)$. Let $\mathscr{A} = \{i\colon (-1)^{n-i} p(x_i) = 1\}$. We show that $\mathscr{A}$ contains at least $n+1$ indices. Suppose that $\mathscr{A}$ contains less than $n+1$ indices. Then there is an $h \in \Pi_n$ such that $h(x) = \operatorname{sgn} p(x)$ and $h(x_i) = -(-1)^{n-i}$ for $i \in \mathscr{A}$. For $\varepsilon > 0$, let $p_\varepsilon = p + \varepsilon h$. For $i \in \mathscr{A}$,

$$(-1)^{n-i} p_\varepsilon(x_i) = 1 - \varepsilon < 1,$$

and for $i \notin \mathscr{A}$,

$$(-1)^{n-i} p_\varepsilon(x_i) = (-1)^{n-i} p(x_i) - \varepsilon(-1)^{n-i} h(x_i) < 1$$

for ε sufficiently small. Thus $p_\varepsilon \in Q_t$ for ε sufficiently small. Furthermore,

$$|p_\varepsilon(x)| = |p(x)| + \varepsilon > M(t, x),$$

which is a contradiction. Thus $\mathscr{A}$ contains at least $n+1$ indices, and as a result, p is q or one of the $q_j(t, \cdot)$, $j = 0,..., n$. Hence, $M(t, x) = \max\{|q(x)|; |q_j(t, x)|, j = 0,..., n\}$.

The next lemma presents an asymptotic estimate for $M(t, x)$.

LEMMA 3. *Let $0 \leqslant x_0 < x_1 < \cdots < x_n$ be fixed. Then there are positive numbers A, X, and T such that $M(t, x) \leqslant Ax^n$ for all $t \geqslant T$ and $x \geqslant X$.*

Proof. Let $q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$. Since q has n zeros and is not identically zero, $\partial q = n$. In fact, $a_n > 0$. Select $X_1 > 0$ such that $x \geqslant X_1$ implies

$$a_n x^n/2 < a_n x^n + \cdots + a_0.$$

Now let $q_j(t, x) = a_n^j(t) x^n + a_{n-1}^j(t) x^{n-1} + \cdots + a_0^j(t)$, $j = 0,..., n$. By the interpolatory conditions defining $q_j(t, \cdot)$

$$\begin{bmatrix} 1 & t & \cdots & t^n \\ 1 & x_0 & \cdots & x_0^n \\ \vdots & \vdots & & \vdots \\ 1 & x_{j-1} & \cdots & x_{j-1}^n \\ 1 & x_{j+1} & \cdots & x_{j+1}^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0^j(t) \\ a_1^j(t) \\ \vdots \\ a_n^j(t) \end{bmatrix} = \begin{bmatrix} -1 \\ (-1)^n \\ \vdots \\ (-1)^{n-j+1} \\ (-1)^{n-j-1} \\ \vdots \\ 1 \end{bmatrix}. \tag{2.3}$$

Using Cramer's rule and evaluating all determinants across the first row, $a_k^j(t) = r_k^j(t)/s^j(t)$, where $r_k^j \in \Pi_n$, $k = 0,..., n$, and $s^j \in \Pi_n$, $j = 0,..., n$.

Moreover, the coefficient of t^n in s^j is given by a Vandermonde determinant and therefore is nonzero, and the coefficient of t^n in r_n^j is zero. That is, $\partial s^j = n$ and $\partial r_n^j \leqslant n-1$, $j = 0,\ldots, n$. Thus we may select positive numbers D and T such that $|a_n^j(t)| < a_n/4$ and $|a_k^j(t)| < D$, $k = 0,\ldots, n-1$, $j = 0,\ldots, n$, for all $t \geqslant T$. Finally, choose $X > X_1$ such that

$$a_n x^n/4 + D(x^{n-1} + \cdots + 1) < a_n x^n/2$$

for $x \geqslant X$. Now for $t \geqslant T$ and $x \geqslant X$,

$$|q_j(t, x)| < a_n x^n/4 + D(x^{n-1} + \cdots + 1) < a_n x^n/2 < q(x),$$

$j = 0,\ldots, n$, and thus $M(t, x) = q(x)$. Since $X > 0$ and $\partial q = n$, we may now select $A > 0$ such that $|q(x)| \leqslant Ax^n$ for $x \geqslant X$, and Lemma 3 now follows from Lemma 2.

The final lemma in this section establishes a bound on $M(t, x)$ for t sufficiently large and x bounded.

LEMMA 4. *Let $0 \leqslant x_0 < x_1 < \cdots < x_n$ be fixed. Given $\beta > 0$ there are positive numbers m and T such that $M(t, x) \leqslant m$ for all $t \geqslant T$ and $x \in [0, \beta]$.*

Proof. Using (2.3) and the succeeding argument, there are constants D and T such that $|q_j(t, x)| \leqslant \alpha^n + D(\alpha^{n-1} + \cdots + 1)$ for $x \in [0, \beta]$ and $t \geqslant T$, where $\alpha = \max(1, \beta)$. Letting $m = \max\{\max_{x \in [0, \beta]} |q(x)|, \alpha^n + D(\alpha^{n-1} + \cdots + 1)\}$, Lemma 4 follows.

3. STRONG UNIQUENESS WHEN $\partial p_f = n-1$

In this section, we show that if $\partial p_f = n-1$, then $1/p_f$ is a strongly unique best approximation to f from R_n. It will then follow that $1/p_f$ satisfies a point Lipschitz condition at f in this case.

THEOREM 5. *Let $f \in C_0^+[0, \infty) \backslash R_n$ and let $1/p_f$ be the best uniform approximation to f from R_n. If $\partial p_f = n-1$, then there is a constant $\gamma_f > 0$ such that*

$$\|f - 1/p\| \geqslant \|f - 1/p_f\| + \gamma_f \|1/p - 1/p_f\| \tag{3.1}$$

for all $1/p \in R_n$.

Proof. Suppose there is no $\gamma_f > 0$ such that (3.1) is valid for all $1/p \in R_n$. Then there is a sequence $\{1/p_k\}$ in R_n such that

$$\gamma_k = \frac{\|f - 1/p_k\| - \|f - 1/p_f\|}{\|1/p_k - 1/p_f\|} \to 0 \tag{3.2}$$

as $k \to \infty$. The sequence $\{\|1/p_k\|\}$ is bounded. Otherwise, for a subsequence $\{1/p_\nu\}$, we would have $\|f - 1/p_\nu\| \to \infty$ and

$$\gamma_\nu \geqslant \frac{\|f - 1/p_\nu\| - \|f - 1/p_f\|}{\|f - 1/p_\nu\| + \|f - 1/p_f\|} \to 1$$

as $\nu \to \infty$, which is contrary to (3.2). Let $\|f - 1/p_k\| \leqslant M$ for all k. Then

$$0 \leqslant \frac{\|f - 1/p_k\| - \|f - 1/p_f\|}{M + \|f - 1/p_f\|} \leqslant \gamma_k$$

and by (3.2), $\lim_{k\to\infty} \|f - 1/p_k\| = \|f - 1/p_f\|$. By the proof of Theorem 4 in Dunham and Taylor [5], $p_k \to p_f$ as $k \to \infty$ in the sense of coefficient convergence.

Since $\partial p_f = n - 1$, $1/p_f$ is the best uniform approximation to f from R_{n-1} and, by Brink's result [1], is strongly unique relative to R_{n-1}. By (3.2) we may then assume that $\partial p_k = n$ for all k. Since $\partial p_k > \partial p_f$ and $p_k(x) > 0$ and $p_f(x) > 0$ for $x \in [0, \infty)$, there is a $t_k > 0$ such that $p_k(x) > p_f(x)$ for all $x \geqslant t_k$. In addition, we choose t_k so that

$$t_k \to \infty \tag{3.3}$$

as $k \to \infty$. Whether $f - 1/p_f$ demonstrates the standard or the nonstandard alternation in Theorem 1, there are $n + 1$ points $0 \leqslant x_0 < \cdots < x_n$ such that

$$(-1)^{n-i}(f(x_i) - 1/p_f(x_i)) = \|f - 1/p_f\|, \tag{3.4}$$

$i = 0,..., n$. For $i = 0,..., n$,

$$(-1)^{n-i}(f(x_i) - 1/p_k(x_i)) \leqslant \|f - 1/p_k\|, \tag{3.5}$$

$i = 0,..., n$. Subtracting (3.4) from (3.5) and multiplying by $p_f(x_i)\, p_k(x_i)$, we obtain

$$(-1)^{n-i}(p_k(x_i) - p_f(x_i)) \leqslant K\Delta_k, \tag{3.6}$$

where $\Delta_k = \|f - 1/p_k\| - \|f - 1/p_f\|$ and $K = \sup\{p_f(x_i)\, p_k(x_i)\colon\ i = 0,...,n,$ $k = 1, 2,...\}$. Furthermore,

$$p_k(t_k) - p_f(t_k) > 0 > -K\Delta_k. \tag{3.7}$$

By (2.1), (2.2), (3.6), and (3.7),

$$|p_k(x) - p_f(x)| \leqslant K\Delta_k M(t_k, x) \tag{3.8}$$

for all $x \in [0, \infty)$ and $k = 1, 2, \ldots$. By Lemma 3 select positive constants A, X, and T such that

$$M(t, x) \leqslant Ax^n \tag{3.9}$$

for $x \geqslant X$ and $t \geqslant T$. Let $p_f(x) = a_{n-1}x^{n-1} + \cdots + a_0$, where $a_{n-1} > 0$, and $p_k(x) = a_n^k x^n + a_{n-1}^k x^{n-1} + \cdots + a_0^k$, where $a_n^k > 0$ and $a_n^k \to 0$ and $a_j^k \to a_j$, $j = 0, \ldots, n-1$, as $k \to \infty$. Select $\beta > X$ such that

$$a_{n-1} - \frac{|a_{n-2}| + 1}{x} - \cdots - \frac{|a_0| + 1}{x^{n-1}} > \frac{a_{n-1}}{2}$$

for $x \geqslant \beta$. Now by (3.3), we may select k_0 such that for $k \geqslant k_0$, $t_k \geqslant T$, $a_{n-1}^k > 3a_{n-1}/4$, and $|a_j^k - a_j| < 1$, $j = 0, \ldots, n-2$. Thus for $x \geqslant \beta$ and $k \geqslant k_0$, $|a_j^k| < |a_j| + 1$, $j = 0, \ldots, n-2$,

$$\begin{aligned} p_f(x) &\geqslant x^{n-1}\left(a_{n-1} - \frac{|a_{n-2}| + 1}{x} - \cdots - \frac{|a_0| + 1}{x^{n-1}}\right) \\ &> a_{n-1}x^{n-1}/2 \end{aligned}$$

and

$$\begin{aligned} p_k(x) &\geqslant a_{n-1}^k x^{n-1} + \cdots + a_0^k \\ &\geqslant x^{n-1}\left(\frac{3a_{n-1}}{4} - \frac{|a_{n-2}| + 1}{x} - \cdots - \frac{|a_0| + 1}{x^{n-1}}\right) > a_{n-1}x^{n-1}/4. \end{aligned}$$

Thus by (3.9)

$$\begin{aligned} \left|\frac{1}{p_f(x)} - \frac{1}{p_k(x)}\right| &= \frac{|p_k(x) - p_f(x)|}{p_f(x)\, p_k(x)} \\ &\leqslant \left(\frac{8AKx^{2-n}}{a_{n-1}^2}\right)\Delta_k \leqslant \left(\frac{8AK\beta^{2-n}}{a_{n-1}^2}\right)\Delta_k \end{aligned} \tag{3.10}$$

for $x \geqslant \beta$, $k \geqslant k_0$, and $n \geqslant 2$. If $n = 1$, then $\partial p_f = 1$, and the case under consideration does not apply (see [5]).

Since $p_f(x) > 0$ for $x \in [0, \beta]$ and $p_k \to p_f$ uniformly on $[0, \beta]$ as $k \to \infty$, there are numbers $s > 0$ and $k_1 \geqslant k_0$ such that $p_f(x) \geqslant s$ and $p_k(x) \geqslant s$ for $x \in [0, \beta]$ and $k \geqslant k_1$. By Lemma 4 and (3.3) there are numbers $m > 0$ and $k_2 \geqslant k_1$ such that $M(t_k, x) \leqslant m$ for all $x \in [0, \beta]$ and $k \geqslant k_2$. Thus if $k \geqslant k_2$, then by (3.8)

$$\left|\frac{1}{p_f(x)} - \frac{1}{p_k(x)}\right| = \frac{|p_k(x) - p_f(x)|}{p_f(x)\, p_k(x)} \leqslant \frac{mK\Delta_k}{s^2} \tag{3.11}$$

for $x \in [0, \beta]$. By (3.10) and (3.11)

$$\|1/p_k - 1/p_f\| \leqslant M(\|f - 1/p_k\| - \|f - 1/p_f\|)$$

for $k \geqslant k_2$, where $M = \max\{8AK\beta^{2-n}/a_{n-1}^2, mK/s^2\}$. This contradicts (3.2), and Theorem 5 is proven.

We conclude this note with the companion point Lipschitz result. The proof is identical to the proof of the theorem on p. 82 of Cheney [3] with $\lambda_f = 2/\gamma_f$ and is omitted.

THEOREM 6. *Let $f \in C_0^+[0, \infty)\backslash R_n$ and let $1/p_f$ be the best uniform approximation to f from R_n. Then there is a constant $\lambda_f > 0$ such that*

$$\|1/p_g - 1/p_f\| \leqslant \lambda_f \|g - f\|$$

for all $g \in C_0^+[0, \infty)$, where $1/p_g$ denotes the best uniform approximation to g from R_n.

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